Perturbed solitons in birefringent fibres

2003 J. Phys. A: Math. Gen. 36 4841

(http://iopscience.iop.org/0305-4470/36/17/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4841–4848 PII: S0305-4470(03)53632-0

Perturbed solitons in birefringent fibres

Theodoros P Horikis and John N Elgin

Department of Mathematics, Imperial College, London SW7 2BZ, UK

E-mail: theodore.horikis@ic.ac.uk and j.elgin@ic.ac.uk

Received 16 September 2002, in final form 18 February 2003
Published 16 April 2003
Online at stacks.iop.org/JPhysA/36/4841

Abstract
A perturbation theory based on inverse scattering theory is developed for the coupled nonlinear Schrödinger equations. The theory finds useful application to the study of pulse propagation down a birefringent optical fibre and is used to examine features of the radiation field shed by a soliton pulse propagating down the fibre. The radiation field is linked to the scattering data through a transform pair which in the linear limit reduces to the forward and inverse Fourier transform pair. A complementary approach, which is in total agreement to these results, is also discussed.

PACS numbers: 02.30.Ik, 02.30.Uu, 42.25.Lc, 42.65.Tg, 42.81.Dp

1. Introduction

When an ultrashort pulse propagates down an anomalously dispersive birefringent optical fibre, complex features develop which require explanation. The object of this paper is to outline a formalism developed within the framework of inverse scattering theory based on the Manakov system, which admits useful application to the study of many of these observed features. One such feature is polarization mode dispersion (PMD) which is one of the most important considerations in transmission systems. A second, the subject of this paper, concerns the properties of the radiation field shed by the soliton pulse when birefringence is present in the fibre.

The theory developed here is a direct extension of one previously published which has application to the isotropic, i.e. nonbirefringent, case [1]. An earlier study on this problem [2] is based on a perturbation theory obtained from a direct linearization of the Manakov equations. This is a complementary approach to the one described here but is one which does not, we contend, use the best mathematical framework; that is one based on inverse scattering theory in which a 'potential' (the complex envelope of the optical pulse) and associated scattering data are spectral transforms of one another; see equations (5) and (6). In addition, the present work completes the analysis presented in [3] where the adiabatic change of the vector soliton parameters, in the presence of perturbation terms, was calculated.

Ultrashort pulse propagation down an anomalously dispersive, birefringent optical fibre is described by the vector nonlinear Schrödinger equation (VNLS)

$$\mathrm{i}\boldsymbol{q}_x + \mathrm{i}\mu\sigma_3\boldsymbol{q}_t - \boldsymbol{q}_{tt} - 2\boldsymbol{q}^\dagger\boldsymbol{q}\boldsymbol{q} = \mathbf{0}. \tag{1}$$

Here, a suffix denotes a partial derivative, † denotes Hermitian conjugation and $\mu = O(\epsilon)$ is the 'small' birefringence parameter. Throughout bold letters denote column vectors, so that $\boldsymbol{q} = (q_1, q_2)^{\mathrm{T}}$. The roles of the independent variables x and t are such that 'time' x is distance propagated by the pulse down the fibre, 'spatial' coordinate t is a retarded time variable indicating position along the pulse, while σ_3 is the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Equation (1) is the lowest order nontrivial amplitude equation obtained from a multiple scales analysis of Maxwell's equations as appropriate to the fibre optic problem. Both dispersion and nonlinearity are present, where the latter is the Kerr nonlinearity which corresponds to the intensity dependence of the refractive index of the host medium. With μ set to zero the VNLS equation is known to be integrable using the techniques of inverse scattering theory [4]. In particular, it has the single soliton solution

$$\boldsymbol{q} \equiv \boldsymbol{q}_s = q_s \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{3}$$

where (scalar) q_s is defined by

$$q_s = 2\eta_1 \exp\left(-2\mathrm{i}\xi_1 t + 4\mathrm{i}\left(\xi_1^2 - \eta_1^2\right)x\right)\operatorname{sech}(2\eta_1(t - 4\xi_1 x)). \tag{4}$$

The solution (3) is hereafter denoted as $\boldsymbol{q}_s$, the vector soliton. The parameters ξ_1, η_1, characterize the soliton, while θ is the projection angle of the pulse onto each (linear) polarization mode. Permitting μ to be nonzero will modify the solution (3) in several distinct ways; these include possible changes in the solitonic parameters ξ_1 and η_1, the generation of a soliton shadow and the generation of a radiation field in each polarization mode. In this paper we will only be interested in the generation of the radiation field. Indeed, perturbations invariably generate a background radiation field, which is superimposed on the soliton pulse. In a strongly birefringent fibre, the radiation will emanate from the soliton at different characteristic speeds in the two polarization modes.

The main results of this paper are divided into two sections. In section 2 we use results from perturbation theory on the Manakov system to derive a nonlinear transform which links the scattering data and the radiation field. In section 3 a complementary approach is introduced, which makes use of the fact that the complex envelope of the radiation field and the scattering data are spectral transforms of one another. A simple transformation is first used to remove the birefringence term from equation (1), effectively changing an input soliton profile $\boldsymbol{q}_s(0, t)$, equation (3), into a mixture of soliton and radiation: all features of birefringence within the fibre are now transformed to a set of suitable initial conditions. The two approaches discussed, respectively, in sections 2 and 3 complement one another in the following sense: the method described in section 2 is best for a study of the initial evolution of the radiation field, while that in section 3 is best suited to describe the asymptote field.

2. The transform pair

As for a similar study of the scalar problem [1], the advantages of using a natural mathematical framework based on inverse scattering theory is emphasized. The spectral transform is a mapping from a potential $\boldsymbol{q}(x, t)$ into a set of scattering data $S_{ij}(x, \zeta)$, $i, j = 1, 2, 3$, where ζ

is an eigenparameter. The inverse transform permits construction of the 'potential' $\boldsymbol{q}$ from a limited set of the data S_{ij}, namely [5]

$$S_{ij} = \int_{-\infty}^{+\infty} \boldsymbol{\phi}^{(j)} \wedge \hat{\boldsymbol{\psi}}^{(i)} \begin{pmatrix} \boldsymbol{q} \\ -\boldsymbol{q}^* \end{pmatrix} \mathrm{d}t \tag{5}$$

with an inverse

$$\begin{pmatrix} \boldsymbol{q} \\ -\boldsymbol{q}^* \end{pmatrix} = \frac{1}{\pi} \int_C \left(\frac{S_{21}}{S_{11}} \boldsymbol{\psi}^{(2)} \vee \hat{\boldsymbol{\psi}}^{(1)} + \frac{S_{31}}{S_{11}} \boldsymbol{\psi}^{(3)} \vee \hat{\boldsymbol{\psi}}^{(1)} \right) \mathrm{d}\zeta \\ - \frac{1}{\pi} \int_{\bar{C}} \left(\frac{\Delta_{21}}{\Delta_{11}} \boldsymbol{\psi}^{(1)} \vee \hat{\boldsymbol{\psi}}^{(2)} + \frac{\Delta_{31}}{\Delta_{11}} \boldsymbol{\psi}^{(1)} \vee \hat{\boldsymbol{\psi}}^{(3)} \right) \mathrm{d}\zeta. \tag{6}$$

Here, $\boldsymbol{\phi}^{(i)} \wedge \hat{\boldsymbol{\psi}}^{(j)}$ and $\boldsymbol{\psi}^{(i)} \vee \hat{\boldsymbol{\psi}}^{(j)}$ are four component row and column vectors, respectively, whose components are made of products between Jost function components for the forward and adjoint scattering problems. Namely $\boldsymbol{\phi}^{(i)} \wedge \hat{\boldsymbol{\psi}}^{(j)} = \left(\phi_2^{(i)} \hat{\psi}_1^{(j)}, \phi_3^{(i)} \hat{\psi}_1^{(j)}, \phi_1^{(i)} \hat{\psi}_2^{(j)}, \phi_1^{(i)} \hat{\psi}_3^{(j)}\right)$ and $\boldsymbol{\psi}^{(i)} \vee \hat{\boldsymbol{\psi}}^{(j)} = \left(\psi_1^{(i)} \hat{\psi}_2^{(j)}, \psi_1^{(i)} \hat{\psi}_3^{(j)}, -\psi_2^{(i)} \hat{\psi}_1^{(j)}, -\psi_3^{(i)} \hat{\psi}_1^{(j)}\right)^{\mathrm{T}}$. Specific forms for those components are quoted in the appendix. The quantities Δ_{ij} are cofactors of the matrix elements S_{ij}, while $C(\bar{C})$ is a contour running from $-\infty + \mathrm{i}\epsilon$ $(-\infty - \mathrm{i}\epsilon)$ to $+\infty + \mathrm{i}\epsilon$ $(+\infty - \mathrm{i}\epsilon)$ passing above (below) all zeros of S_{11} (Δ_{11}) ; see [5] for further details.

We will consider equation (1) in the form

$$\mathrm{i}\boldsymbol{q}_x - \boldsymbol{q}_{tt} - 2\boldsymbol{q}^{\dagger}\boldsymbol{q}\boldsymbol{q} = \mathrm{i}\boldsymbol{F} \tag{7}$$

where $\boldsymbol{F} = -\mu\sigma_3\boldsymbol{q}_t$. The scattering data, S_{ij}, evolve according to equation [5]

$$S_{ij,x} = S_{ij,x}^{(0)} + \int_{-\infty}^{+\infty} \boldsymbol{\phi}^{(j)} \wedge \boldsymbol{\psi}^{(i)} \begin{pmatrix} \boldsymbol{F} \\ -\boldsymbol{F}^* \end{pmatrix} \mathrm{d}t \tag{8}$$

where $S_{ij,x}^{(0)}$ represents the evolution of S_{ij} for the unperturbed system. In particular, $S_{i1,x}^{(0)} = -4\mathrm{i}\zeta^2 S_{i1}^{(0)}, i = 2, 3$, while $S_{11,x}^{(0)} = 0$. We are interested in the case when a single soliton $\boldsymbol{q}_s$ accompanied by radiation is present in the fibre; then, we write

$$\boldsymbol{q}(x, t) = \boldsymbol{q}_s(x, t) + \boldsymbol{\delta q}(x, t) \tag{9}$$

where $\boldsymbol{\delta q} = (\delta q_1, \delta q_2)^{\mathrm{T}}$. On substituting in equations (5) and (6) we obtain the transform pair linking the scattering data and the radiation field, namely

$$\begin{pmatrix} \delta q_1 \\ \delta q_2 \end{pmatrix} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left(\frac{S_{21}}{S_{11}} \begin{pmatrix} \psi_1^{(2)} \hat{\psi}_2^{(1)} \\ \psi_1^{(2)} \hat{\psi}_3^{(1)} \end{pmatrix} + \frac{S_{31}}{S_{11}} \begin{pmatrix} \psi_1^{(3)} \hat{\psi}_2^{(1)} \\ \psi_1^{(3)} \hat{\psi}_3^{(1)} \end{pmatrix} \right) \mathrm{d}\xi \\ - \frac{1}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\Delta_{21}}{\Delta_{11}} \begin{pmatrix} \psi_1^{(1)} \hat{\psi}_2^{(2)} \\ \psi_1^{(1)} \hat{\psi}_3^{(2)} \end{pmatrix} + \frac{\Delta_{31}}{\Delta_{11}} \begin{pmatrix} \psi_1^{(1)} \hat{\psi}_2^{(3)} \\ \psi_1^{(1)} \hat{\psi}_3^{(3)} \end{pmatrix} \right) \mathrm{d}\xi \tag{10}$$

and

$$S_{ij} = \int_{-\infty}^{+\infty} \boldsymbol{\phi}^{(j)} \wedge \hat{\boldsymbol{\psi}}^{(i)} \begin{pmatrix} \boldsymbol{\delta q} \\ -\boldsymbol{\delta q}^* \end{pmatrix} \mathrm{d}t \qquad (i, j) = (2, 1), (3, 1). \tag{11}$$

If the input pulse to the fibre is the soliton $\boldsymbol{q}_s(0, t)$ then the initial condition for $S_{ij}(0, \zeta)$ is $S_{ij} = 0, i \neq j$. Further, if the perturbing term $\boldsymbol{F}$ is dispersive, and small, $O(\epsilon)$ say, the change in the soliton parameters $\xi_1(x)$ and $\eta_1(x)$ is $O(\epsilon^2)$, which are ignored here since the perturbation theory to be developed is first order, $O(\epsilon)$. In the absence of $\boldsymbol{F}$, $S_{ij}, i \neq j$ would remain zero for all x; with the perturbation present we obtain $O(\epsilon)$ expressions for $S_{ij}, i \neq j$, which are valid for distances x up to $\epsilon x \ll 1$.

Note that equations (10) and (11) are the direct extension of the application of the Fourier transform to linear systems, as appropriate to the integrable VNLS equation. Indeed, in the limit where the pulse $\boldsymbol{q}(x,t)$ has no soliton component and simply represents a weak radiation field, $\boldsymbol{\delta q}(x,t)$ say, these reduce to

$$\begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix} = -\int_{-\infty}^{+\infty} \begin{pmatrix} \delta q_1^* \\ \delta q_2^* \end{pmatrix} \exp(-\mathrm{i}\omega t)\,\mathrm{d}t \tag{12}$$

$$\begin{pmatrix} \delta q_1^* \\ \delta q_2^* \end{pmatrix} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix} \exp(\mathrm{i}\omega t)\,\mathrm{d}t \tag{13}$$

where * denotes complex conjugation, and $\omega = 2\xi = 2\mathrm{Re}\{\zeta\}$. Each component $\delta q_1^*, \delta q_2^*$ of $\boldsymbol{\delta q}^*$ is here linked to the one piece of scattering data, S_{21} and S_{31}, respectively. This simplifying feature is lost for the full (nonlinear) system.

Evaluating the integrals in equation (8) produces

$$\begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix}_x = -4\mathrm{i}\zeta^2 \begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix} - 2\mathrm{i}\zeta\mu \begin{pmatrix} S_{21} \\ -S_{31} \end{pmatrix} + \mathrm{i}\mu\sin(2\theta)(\zeta - \mathrm{i}\eta_1)\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}\hat{q}_s^*. \tag{14}$$

We will let $\zeta = \xi$ to generate the continuum (radiation) field. Further, $\hat{q}_s^*$ is the conjugate of the Fourier transform of the scalar soliton amplitude q_s, namely

$$\hat{q}_s(x,\zeta) = \pi\exp\left(-4\mathrm{i}\eta_1^2 x\right)\mathrm{sech}(\pi\zeta/2\eta_1). \tag{15}$$

In the *linearized* problem, the vector $(S_{21}, S_{31})^{\mathrm{T}}$ has the same polarization state as the pulse $\hat{\boldsymbol{q}}_s$. For the nonlinear problem equation (14) indicates that $(S_{21}, S_{31})^{\mathrm{T}}$ is generated in a polarization state which is *orthogonal* to $\hat{\boldsymbol{q}}_s$; this aspect will be discussed further below. This equation also indicates that the inhomogeneous term vanishes whenever $\theta = 0$ or $\pi/2$, as should be the case since the vector problem then reduces to the scalar case in one or other polarization state with soliton input, where no radiation is expected.

The term $-2\mathrm{i}\zeta\mu(S_{21}, -S_{31})$ is precisely the additional term that is required to ensure that S_{21} and S_{31} follow their respective characteristics. However, since S_{ij} are $O(\mu)$ it follows that this term in equation (14) is second order in the small parameter μ, and hence violates our assumption of a first-order perturbation theory. We will ignore this inconsistency since the equation has all the desired features for the generation of S_{21} and S_{31}: the dispersive term in the parameter ζ^2, the requirement that S_{21} and S_{31} evolve along separate characteristics as appropriate for a birefringent fibre and more interestingly the fact that the radiation field (or rather, the vector $(S_{21}, S_{31})^{\mathrm{T}}$) is generated in a polarization state that is orthogonal to the polarization state of the soliton.

Once S_{21} and S_{31} are known, determination of δq_1 and δq_2 requires the further evaluation of the integrals (10): we now show how the latter can be changed into a simpler algorithmic step. The evolution of the spectral data is now governed by equation (14), subject to the initial condition that $S_{21}(0,\xi) = S_{31}(0,\xi) = 0$. As for the scalar problem [1], it is useful to introduce two quantities related to S_{21} and S_{31}, namely, the *associate fields* $f_1(x,t)$ and $f_2(x,t)$. Define

$$\hat{f}_1(x,\xi) = \frac{S_{21}^*(x,\xi)}{4\xi^2+1} \qquad \hat{f}_2(x,\xi) = \frac{S_{31}^*(x,\xi)}{4\xi^2+1}$$

where $\hat{f}(x,\xi) \equiv \mathcal{F}\{f(x,t)\} = \int_{-\infty}^{+\infty}\exp(-2\mathrm{i}\xi t)f(x,t)\,\mathrm{d}t$ is the Fourier transforms of $f(x,t)$. Then, equation (14) becomes

$$-\mathrm{i}\begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \end{pmatrix}_x = 4\xi^2\begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \end{pmatrix} + 2\xi\mu\begin{pmatrix} \hat{f}_1 \\ -\hat{f}_2 \end{pmatrix} + \frac{2\mu\sin(2\theta)}{2\xi+\mathrm{i}}\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}\hat{q}_s^*$$

or in t-space

$$-\mathrm{i}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_x = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_{tt} + \mathrm{i}\mu\begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}_t - \frac{\mathrm{i}\mu}{2}\sin(2\theta)\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} q_s \otimes h \tag{16}$$

where q_s is the (scalar) soliton expression, equation (4),

$$h(t) = \begin{cases} \exp(t) & t < 0 \\ 0 & t > 0 \end{cases}$$

and $\otimes$ denotes convolution product. From equation (10), using solitonic expressions for $\psi_j^{(\mathrm{i})}$, and for $S_{11}(\xi) = (2\xi - \mathrm{i})/(2\xi + \mathrm{i}) = \Delta_{11}^*(\xi)$, and then evaluating the various integrals, the following expressions are obtained for δq_1 and δq_2

$$-\delta q_1 = (M - N\sin^2\theta) f_1 + \tfrac{1}{2}\sin(2\theta) N f_2 - q_s^2 \cos\theta (f_1^* \cos\theta + f_2^* \sin\theta) \tag{17a}$$

$$-\delta q_2 = (M - N\cos^2\theta) f_2 + \tfrac{1}{2}\sin(2\theta) N f_1 - q_s^2 \sin\theta (f_2^* \sin\theta + f_1^* \cos\theta). \tag{17b}$$

To make these awkward expressions more manageable, we have introduced the operators

$$M = \frac{\partial^2}{\partial t^2} - 2\tanh t \frac{\partial}{\partial t} + \tanh^2 t \qquad N = (1 - \tanh t)\frac{\partial}{\partial t} + \tanh^2 t - \tanh t.$$

The algorithm for finding δq_1 and δq_2 is first to solve equations (16) for $f_1(x, t)$ and $f_2(x, t)$ subject to the initial condition that $f_1(0, t)$ and $f_2(0, t)$ are both zero (so that the limitations of the method derived in [1] do not apply here). This is now straightforward since both f_1 and f_2 satisfy *linear* differential equations, and can be easily obtained using standard (Fourier) transform methods. We find δq_1 and δq_2 simply by using equations (17). This is, again, relatively straightforward requiring only differentiation of the known functions f_1 and f_2.

The qualitative features of equations (16) are straightforward: dispersive radiation is generated, which then propagates along the characteristics $x \pm \mu t$. Both these contribute to the generation of both δq_1 and δq_2, in accordance with equations (17). Near the soliton, δq_1 and δq_2 have a complicated structure with no readily discernable features. Away from the soliton—that is at large values of $|t|$—we expect the radiation field to evolve in accordance with the linear theory: a predominance of δq_1 should appear in the slow polarization mode, δq_2 in the fast, with each field propagating away from the (source) soliton pulse at a group velocity determined by the frequency shifts $\delta\omega = \pm\mu/2$. At large values of $|t|$, the cross terms proportional to q_s^2 can be ignored in equations (17), and we may approximate $\tanh t \simeq \pm 1$ as appropriate. Then,

$$M \simeq \left(\frac{\partial}{\partial t} \mp 1\right)^2 \qquad t \to \pm\infty$$

while

$$N \simeq \begin{cases} 0 & t \to \infty \\ 2\left(\frac{\partial}{\partial t} + 1\right) & t \to -\infty. \end{cases}$$

Hence, as $t \to +\infty$,

$$\begin{pmatrix} \delta q_1 \\ \delta q_2 \end{pmatrix} \simeq \left(\frac{\partial^2}{\partial t^2} - 2\frac{\partial}{\partial t} + 1\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{18}$$

and as $t \to -\infty$

$$\begin{pmatrix} \delta q_1 \\ \delta q_2 \end{pmatrix} \simeq \left(\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t} + 1\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} 2\left(\frac{\partial}{\partial t} + 1\right) f_\perp. \tag{19}$$

Here we have an interesting asymmetry with no ready explanation. For large values of the parameter μ (let us assume, for the moment, that the perturbation theory continues to hold), one expects f_1 to dominate f_2 as $t \to +\infty$, since the characteristic for f_1 is $t - \mu x$, and hence we expect δq_1 to dominate δq_2; this would be in accord with simple intuition. The same intuition—with f_2 now dominating f_1—fails at $t \to -\infty$ because of the presence of $f_\perp = -f_1 \sin\theta + f_2 \cos\theta$ in equation (19); here, now, (large) f_2 will also contribute to δq_1. If the latter terms were missing, the other difference between equations (18) and (19) can be explained in terms of the phase shift induced by the presence of the soliton pulse, i.e. $(\partial_t - 1)^2/(\partial t + 1)^2 \to (\omega + \mathrm{i})^2/(\omega - \mathrm{i})^2$ in frequency space, which is the phase shift experienced by a linear plane wave $\exp(\mathrm{i}\omega t)$ on passing from $t \to +\infty$ to $t \to -\infty$ through a soliton pulse [6].

3. Removal of the birefringence term

We now describe a complementary approach to that given in the previous section which makes direct use of the spectral transform, equations (12) and (13). The model equation is now the unperturbed VNLS and aspects of the radiation field appear in nonzero expressions for $S_{21}(0, \xi)$, $S_{31}(0, \xi)$. The transformation

$$\boldsymbol{p} = \exp(-\mathrm{i}\mu\sigma_3 t/2 - \mathrm{i}\mu^2 x/4)\boldsymbol{q} \tag{20}$$

removes the birefringent term from equation (1), producing the Manakov evolution equation for $\boldsymbol{p}(x, t)$

$$\mathrm{i}\boldsymbol{p}_x - \boldsymbol{p}_{tt} - 2\boldsymbol{p}^\dagger \boldsymbol{p}\boldsymbol{p} = \boldsymbol{0}. \tag{21}$$

This has the soliton solution equation (3), which we temporarily denote as $\boldsymbol{p}_s$. The corresponding soliton solution for $\boldsymbol{q}(x, t)$, which we denote as $\boldsymbol{q}_s^{(\mu)}$, is obtained by inverting transformation (20), to yield

$$\boldsymbol{q}_s^{(\mu)}(x, t) = \exp(\mathrm{i}\mu^2 x/4) \begin{pmatrix} \exp(\mathrm{i}\mu t/2)\cos\theta \\ \exp(-\mathrm{i}\mu t/2)\sin\theta \end{pmatrix} q_s \tag{22}$$

with scalar q_s defined in equation (4). Note the spectral splitting of the two polarization components of $\boldsymbol{q}_s^{(\mu)}$, yielding peaks displaced from the origin at $\delta\omega = \pm\mu/2$. A soliton pulse $\boldsymbol{q}_s^{(\mu)}$ inserted into the birefringent fibre at $x = 0$ will continue to propagate as a soliton, with no change to the structure other than that incorporated into the usual soliton phase shift characterized by the leading exponential term in equation (22). In practice, it is rarely feasible to tailor such input pulses; rather, a typical input is $\boldsymbol{q}_s^{(\mu=0)}(0, t) \equiv \boldsymbol{q}_s(0, t)$. In consequence, the soliton input to the fibre is now accompanied by a radiation field (we hereafter set $2\eta_1 = 1$),

$$\boldsymbol{\delta q}(0, t) = \begin{pmatrix} (1 - \exp(\mathrm{i}\mu t/2))\cos\theta \\ (1 - \exp(-\mathrm{i}\mu t/2))\sin\theta \end{pmatrix} \operatorname{sech} t. \tag{23}$$

This is now inserted into equation (11), together with the solitonic expressions for $\boldsymbol{\phi}^{(1)} \wedge \hat{\boldsymbol{\psi}}^{(i)}$, $i = 1, 2$ and the integrals evaluated to determine $S_{21}(0, \omega)$, $S_{31}(0, \omega)$: find

$$\begin{pmatrix} S_{21}(\omega) \\ S_{31}(\omega) \end{pmatrix} = \frac{\mu}{4}\sin(2\theta)\left(\frac{\hat{q}_s(\omega - \mu/2)}{\omega - \mu/2 + \mathrm{i}} + \frac{\hat{q}_s(\omega + \mu/2)}{\omega + \mu/2 + \mathrm{i}}\right)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$
$$- \frac{1}{8}\mu^2 \hat{q}_s''(\omega)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \text{h.o.t.} \tag{24}$$

where we have introduced $\omega = 2\xi$, $\hat{q}_s(\omega) = \operatorname{sech}(\pi\omega/2)$ is the Fourier transform of $q_s(0, t)$, and $''$ denotes second derivative w.r.t. ω. Terms of order μ^3 and higher have been neglected, but no expansion has been made for $\hat{q}_s(\omega \pm \mu)/(\omega \pm \mu + \mathrm{i})$. The structure of equation (24) is interesting: the leading $O(\epsilon)$ contribution is polarized orthogonally to the soliton pulse. The two distinct spectral functions have peaks at $\omega = \pm\mu/2$; a simple (Fourier) interpretation of this would suggest that radiation produced by each 'source' would then propagate with the correct group velocity along each characteristic. But of course, the kernels used in the construction of δq_1 and δq_2 from S_{21} and S_{31} are not the simple Fourier kernels, but rather the soliton expressions for the Jost functions that appear in the appendix, so care should be taken with such an interpretation. What one might expect is that at large 'distances' from the soliton pulse, i.e. at large values of $|t|$, a predominance of δq_1 (the slow mode) should be present for large positive values of t, while a predominance of δq_2 should be present at large negative values of t. It is not at all apparent how the information contained in equation (10) with (24) accommodates this expectation. Interestingly, the Fourier transform of $\boldsymbol{\delta q}(0, t)$, equation (23), projected onto the orthogonal polarization mode produces

$$\hat{\delta q}(0, \omega) = \tfrac{1}{2}\sin(2\theta)(\hat{q}(\omega - \mu/2) - \hat{q}_s(\omega + \mu/2)) \tag{25}$$

which differs in structure from the same projection of (24) by the absence of the quantities $\pm\mu/(\omega \pm \mu + \mathrm{i})$. In a recent study of the scalar problem [1], we noted that the spectral transform and Fourier transform of $\boldsymbol{\delta q}(0, t)$ were often proportional to one another; this is obviously not the case here. Note further that the component of $(S_{21}(0, \omega), S_{31}(0, \omega))^{\mathrm{T}}$ polarized parallel to the soliton pulse is $O(\mu^2)$.

Knowing $S_{21}(0, \omega)$ and $S_{31}(0, \omega)$, it is straightforward to deduce $S_{21}(x, \omega)$ and $S_{31}(x, \omega)$; these are $S_{i1}(x, \omega) = \exp(-\mathrm{i}\omega^2 x)S_{i1}(0, \omega)$, $i = 1, 2$. Reconstruction of $\delta q_1(x, t)$ and $\delta q_2(x, t)$ then requires the substitution of $S_{i1}(x, \omega)$ into equation (10), the substitutions $S_{11}(x, \omega) = (\omega - \mathrm{i})/(\omega + \mathrm{i}) = \Delta^*_{11}(x, \omega)$, and an evaluation of the resulting integrals.

4. Final comments

In summary, we have developed a perturbation theory to analyse perturbed forms of the VNLS, as appropriate to studies on pulse propagation down an anomalously dispersive, birefringent optical fibre. The formalism is developed within the framework of inverse scattering theory based on the Manakov system, which we believe to be the 'best' mathematical framework to use. The formalism has been used to examine features of pulse propagation down a birefringent fibre, in particular to examine features of the radiation field 'shed' by the soliton pulse. Applications of this formalism have already been considered in a wide area of nonlinear optics such as to the study of polarization mode dispersion [7], to cases of additional perturbations such as higher order dispersion, and to the soliton shadow [8].

Appendix. The Jost functions

We will list here the components for the Jost functions $\boldsymbol{\psi}^{(i)}$ and $\boldsymbol{\psi}^{(j)}$, together with similar components for the adjoint functions. The adjoint Jost functions are obtained from the relationships $\hat{\phi}^{(i)}_j(\zeta, t) = \phi^{(i)}_j(\zeta, t)^*$, and $\hat{\psi}^{(i)}_j(\zeta, t) = \psi^{(i)}_j(\zeta, t)^*$, where * denotes complex conjugate.

$$\phi^{(1)}_1 = \frac{\exp(-\mathrm{i}\zeta t)}{\zeta + \mathrm{i}\eta_1}(\zeta - \mathrm{i}\eta_1 \tanh(2\eta_1 t))$$

$$\phi^{(1)}_2 = -\frac{\mathrm{i}\eta_1}{\zeta + \mathrm{i}\eta_1}\exp\left(-\mathrm{i}\zeta t + 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\cos\theta$$

$$\phi^{(1)}_3 = -\frac{\mathrm{i}\eta_1}{\zeta + \mathrm{i}\eta_1}\exp\left(-\mathrm{i}\zeta t + 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\sin\theta$$

$$\phi^{(2)}_1 = -\frac{\mathrm{i}\eta_1}{\zeta - \mathrm{i}\eta_1}\exp\left(\mathrm{i}\zeta t - 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\cos\theta$$

$$\phi_2^{(2)} = \frac{\exp(\mathrm{i}\zeta t)}{\zeta - \mathrm{i}\eta_1}(\zeta + \mathrm{i}\eta_1(\cos^2\theta\tanh(2\eta_1 t) - \sin^2\theta))$$

$$\phi_3^{(2)} = \frac{\mathrm{i}\eta_1}{\zeta - \mathrm{i}\eta_1}\exp(\mathrm{i}\zeta t)(1 + \tanh(2\eta_1 t))\sin\theta\cos\theta$$

$$\phi_1^{(3)} = -\frac{\mathrm{i}\eta_1}{\zeta - \mathrm{i}\eta_1}\exp\left(\mathrm{i}\zeta t - 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\sin\theta$$

$$\phi_2^{(3)} = \frac{\mathrm{i}\eta_1}{\zeta - \mathrm{i}\eta_1}\exp(\mathrm{i}\zeta t)(1 + \tanh(2\eta_1 t))\sin\theta\cos\theta$$

$$\phi_3^{(3)} = \frac{\exp(\mathrm{i}\zeta t)}{\zeta - \mathrm{i}\eta_1}(\zeta + \mathrm{i}\eta_1(\sin^2\theta\tanh(2\eta_1 t) - \cos^2\theta))$$

$$\psi_1^{(1)} = \frac{\exp(-\mathrm{i}\zeta t)}{\zeta - \mathrm{i}\eta_1}(\zeta - \mathrm{i}\eta_1\tanh(2\eta_1 t))$$

$$\psi_2^{(1)} = -\frac{\mathrm{i}\eta_1}{\zeta - \mathrm{i}\eta_1}\exp\left(-\mathrm{i}\zeta t + 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\cos\theta$$

$$\psi_3^{(1)} = -\frac{\mathrm{i}\eta_1}{\zeta - \mathrm{i}\eta_1}\exp\left(-\mathrm{i}\zeta t + 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\sin\theta$$

$$\psi_1^{(2)} = -\frac{\mathrm{i}\eta_1}{\zeta + \mathrm{i}\eta_1}\exp\left(\mathrm{i}\zeta t - 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\cos\theta$$

$$\psi_2^{(2)} = \frac{\exp(\mathrm{i}\zeta t)}{\zeta + \mathrm{i}\eta_1}(\zeta + \mathrm{i}\eta_1(\cos^2\theta\tanh(2\eta_1 t) + \sin^2\theta))$$

$$\psi_3^{(2)} = \frac{\mathrm{i}\eta_1}{\zeta + \mathrm{i}\eta_1}\exp(\mathrm{i}\zeta t)(-1 + \tanh(2\eta_1 t))\sin\theta\cos\theta$$

$$\psi_1^{(3)} = -\frac{\mathrm{i}\eta_1}{\zeta + \mathrm{i}\eta_1}\exp\left(\mathrm{i}\zeta t - 4\mathrm{i}\eta_1^2 x\right)\operatorname{sech}(2\eta_1 t)\sin\theta$$

$$\psi_2^{(3)} = \frac{\mathrm{i}\eta_1}{\zeta + \mathrm{i}\eta_1}\exp(\mathrm{i}\zeta t)(-1 + \tanh(2\eta_1 t))\sin\theta\cos\theta$$

$$\psi_3^{(3)} = \frac{\exp(\mathrm{i}\zeta t)}{\zeta + \mathrm{i}\eta_1}(\zeta + \mathrm{i}\eta_1(\sin^2\theta\tanh(2\eta_1 t) + \cos^2\theta)).$$

References

[1] Horikis T P and Elgin J N 2001 *J. Opt. Soc. Am.* B **18** 913–8
[2] Lakoba T I and Kaup D J 1997 *Phys. Rev.* E **56** 6147–65
[3] Midrio M, Wabnitz S and Franco P 1996 *Phys. Rev.* E **54** 5743–51
[4] Manakov S V 1974 *Sov. Phys.-JETP* **8** 248–53
[5] Docksey R G and Elgin J N 2000 *SIAM J. Math. Anal.* **32** 54–79
[6] Gordon J P 1992 *J. Opt. Soc. Am.* **9** 91
[7] Horikis T P and Elgin J N 2002 *Opt. Lett.* **27** 1516
[8] Horikis T P and Elgin J N 2003 *Phys. Rev.* E at press